Supersymmetric Solution of PT-/Non-PT-Symmetric and Non-Hermitian Morse Potential via Hamiltonian Hierarchy Method

Metin Aktaş and Ramazan Sever*
Department of Physics, Middle East Technical University
06531 Ankara, Turkey

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Abstract

Supersymmetric solution of PT-/non-PT-symmetric and non-Hermitian Morse potential is studied to get real and complex-valued energy eigenvalues and corresponding wave functions. Hamiltonian Hierarchy method is used in the calculations.

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*Corresponding author: sever@metu.edu.tr

1 Introduction

PT-symmetric Hamiltonians has acquired much interest in recent years[1, 2, 3]. Bender and Boettcher [1] suggested that a non-Hermitian complex potential with the characteristic of PT-invariance has real energy eigenvalue if PT-symmetry is not spontaneously broken. The other concept for a class of non-Hermitian Hamiltonians is pseudo-Hermiticity. This kind of operators satisfy the similarity transformation $\eta\, \hat{H}\, \eta^{-1} = \hat{H}^{\dagger}$ [3, 4, 5]. PT-invariant operators have been analysed for real and complex spectra by using a variety of techniques such as variational methods [7], numerical approaches [8], semiclassical estimates [9], Fourier analysis [10] and group theoretical approach with the Lie algebra [11]. It is pointed out that PT-invariant complex-valued operators may have real or complex energy eigenvalues [12]. Many authors have studied on PT-symmetric and non-PT-symmetric non-Hermitian potential cases such as flat and step potentials with the framework of SUSYQM [13-15], exponential type potentials [16-21], quasi exactly solvable quartic potentials [22-24], complex Hénon-Heiles potential [25], and therein [26-28]. In the present work, the real and complex-valued bound-state energies of the q-deformed Morse potential are evaluated through the Hamiltonian Hierarchy method [29] by following the framework of PT-symmetric quantum mechanics. This method also known as the factorization method of the Hamiltonian introduced by Schrödinger [30], and later developed by Infeld and Hull [31], It is useful to discover for different potentials with equivalent energy spectra in non-relativistic quantum mechanics. Various aspects has been studied within the formalism of SUSYQM [32]. This paper is arranged as follows: In Sec. II we introduce the Hamiltonian Hierarchy method. In Sec. III we apply the method to solve the Schrödinger equation with PT-symmetric and non-PT-symmetric non-Hermitian forms of the q-deformed Morse potential. In Sec. IV we discuss the results.

2 SUSYQM and Hamiltonian Hierarchy Method

Supersymmetric algebra allows us to write Hamiltonians as [30]

$$H_{\pm} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\pm}(x), \tag{1}$$

where The supersymmetric partner potentials $V_{\pm}(x)$ in terms of the superpotential $W(x)$ are given by

$$V_{\pm}(x) = W^2 \pm \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}. \tag{2}$$

The superpotential has a definition

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \left[\frac{d \ln \Psi_0^{(0)}(x)}{dx} \right], \tag{3}$$

where, $\Psi_0^{(0)}(x)$ denotes the ground state wave function that satisfies the relation

$$\Psi_0^{(0)}(x) = N_0 \ \exp \left[-\frac{\sqrt{2m}}{\hbar} \int^x W(x') \ dx' \right]. \tag{4}$$

The Hamiltonian $H_{\pm}$ can also be written in terms of the bosonic operators A^- and A^+

$$H_{\pm} = A^{\mp} \ A^{\pm}, \tag{5}$$

where

$$A^{\pm} = \pm \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \tag{6}$$

It is remarkable result that the energy eigenvalues of H_- and H_+ are identical except for the ground state. In the case of unbroken supersymmetry, the ground state energy of the Hamiltonian H_- is zero $\left(E_0^{(0)} = 0 \right)$ [30]. In the factorization of the Hamiltonian, the Eqs. (1), (5) and (6) are used respectively. Hence, we obtain

$$\begin{aligned} H_1(x) &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \\ &= (A_1^+ \ A_1^-) + E_1^{(0)}. \end{aligned} \tag{7}$$

Comparing each side of the Eq. (7) term by term, we get the Riccati equation for the superpotential $W_1(x)$

$$W_1^2 - W_1' = \frac{2m}{\hbar^2} \left(V_1(x) - E_1^{(0)} \right). \tag{8}$$

Let us now construct the supersymmetric partner Hamiltonian H_2 as

$$\begin{aligned} H_2(x) &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x) \\ &= \left(A_2^- \ A_2^+ \right) + E_2^{(0)}, \end{aligned} \tag{9}$$

and Riccati equation takes

$$W_2^2 + W_2^{'} = \frac{2m}{\hbar^2}\left(V_2(x) - E_2^{(0)}\right). \tag{10}$$

Similarly, one can write in general the Riccati equation and Hamiltonians by iteration as

$$\begin{aligned} W_n^2 \pm W_n^{'} &= \frac{2m}{\hbar^2}\left(V_n(x) - E_n^{(0)}\right) \\ &= \left(A_n^{\pm}\ A_n^{\mp}\right) + E_n^{(0)}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} H_n(x) &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_n(x) \\ &= A_n^{+}\ A_n^{-} + E_n^{(0)}, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{12}$$

where

$$A_n^{\pm} = \pm\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \frac{d}{dx}\left(\ln \Psi_n^{(0)}(x)\right). \tag{13}$$

Because of the SUSY unbroken case, the partner Hamiltonians satisfy the following expressions [30]

$$E_{n+1}^{(0)} = E_n^{(1)}, \quad with \quad E_0^{(0)} = 0, \quad n = 0, 1, 2, \ldots \tag{14}$$

and also the wave function with the same eigenvalue can be written as [30]

$$\Psi_n^{(1)} = \frac{A^{-}\ \Psi_{n+1}^{(0)}}{\sqrt{E_n^{(0)}}}, \tag{15}$$

with

$$\Psi_{n+1}^{(0)} = \frac{A^{+}\ \Psi_n^{(1)}}{\sqrt{E_n^{(0)}}}. \tag{16}$$

This procedure is known as the hierarchy of Hamiltonians.

3 Calculations

3.1 The General q-deformed Morse case

Let us first consider the generalized Morse potential as [16]

$$V_M(x) = V_1\ e^{-2\ a\ x} - V_2\ e^{-a\ x}. \tag{17}$$

where V_1 and V_2 are real parameters. By comparing the Eq. (17) with the following equation

$$V_M(x) = D(e^{-2\ a\ x} - 2qe^{-a\ x}), \tag{18}$$

we have $V_1 = D$ and $V_2 = 2qD$. Therefore we can construct the hierarchy of Hamiltonians for Schrödinger equation with $\ell = 0$,

$$[-\frac{d^2\Psi}{dx^2} + \mu^2(e^{-2\ a\ x} - 2qe^{-a\ x})]\Psi(x) = \varepsilon\Psi(x), \tag{19}$$

where $\mu^2 = \dfrac{2mV_1}{a^2\hbar^2}$ and $E = \varepsilon\dfrac{a^2\hbar^2}{2m}$. We can also write the Riccati equation as

$$W_1^2 - W_1' + \varepsilon_0^{(1)} = V_1(x). \tag{20}$$

Here $V_1(x)$ is the superpartner of the superpotential $W_1(x)$. Following by ansatz equation, we have

$$W_1(x) = -\mu\ e^{-a\ x} + q\ \delta, \tag{21}$$

and inserting this into the Eq. (20), we get

$$\delta = (\mu - \frac{a}{2q}), \tag{22}$$

with the first ground state energy

$$\varepsilon_0^{(1)} = -q^2(\mu - \frac{a}{2q})^2. \tag{23}$$

In order to construct the other superpartner potential $V_2(x)$, we will solve the equation

$$W_1^2 + W_1' + \varepsilon_0^{(1)} = V_2(x). \tag{24}$$

Then we can find the second member superpotential as

$$W_2(x) = -\mu\ e^{-a\ x} + q\ \kappa. \tag{25}$$

Now, putting this ansatz into the Eq. (20), we get

$$\kappa = (\mu - \frac{3a}{2q}), \tag{26}$$

with

$$\varepsilon_0^{(2)} = -q^2(\mu - \frac{3a}{2q})^2. \tag{27}$$

By similar iterations, one can get the general results

$$W_{n+1}(x) = -\mu\ e^{-ax} + q\left[\mu - \frac{a}{q}(n + \frac{1}{2})\right], \tag{28}$$

$$V_{n+1}(x) = \mu^2(e^{-2ax} - 2qe^{-ax}) + 2na\mu\ e^{-ax}, \tag{29}$$

$$E_{n+1}^{(\ell=0)} = -q^2\left[\mu - \frac{a}{q}(n + \frac{1}{2})\right]^2, \tag{30}$$

and ground state wave function

$$\Psi_0(x) = N\ \exp\{-\widetilde{\mu}e^{-ax} + q\left[\mu - \frac{a}{q}(n + \frac{1}{2})\right]x\}. \tag{31}$$

where we choose $\widetilde{\mu} = a\mu$ and set ($\hbar = 2m = 1$) in Eq. (30).

3.2 Non-PT-symmetric and Non-Hermitian Morse case

Let us now consider the Eq. (17) with respect to $V_1 \rightarrow\ D$ as real and $V_2 \rightarrow\ 2iqD$ as complex parameters. Hence we construct the hierarchy of Hamiltonian of the Schrödinger equation for the complexified Morse potential as

$$[-\frac{d^2\Psi}{dx^2} + \mu^2(e^{-2\ a\ x} - 2iqe^{-a\ x})]\Psi(x) = \varepsilon\Psi(x), \tag{32}$$

where $\mu^2 = \dfrac{2mV_1}{a^2\hbar^2}$ and $E = \varepsilon\dfrac{a^2\hbar^2}{2m}$.

Applying the hierarchy of Hamiltonians as in the previous section, the $(n+1)$-th member results will be

$$W_{n+1}(x) = -\mu\ e^{-a\ x} + q\left[i\mu - \frac{a}{q}(n+\frac{1}{2})\right], \tag{33}$$

$$V_{n+1}(x) = \mu^2(e^{-2ax} - 2qe^{-ax}) + 2na\mu\ e^{-ax}, \tag{34}$$

$$E_{n+1}^{(\ell=0)} = -q^2\left[i\mu - \frac{a}{q}(n+\frac{1}{2})\right]^2, \tag{35}$$

with

$$\Psi_0(x) = N\ \exp\ \{-\widetilde{\mu}e^{-ax} + q\left[i\mu - \frac{a}{q}(n+\frac{1}{2})\right]x\}, \tag{36}$$

where $\widetilde{\mu} = a\mu$.

3.3 PT-symmetric and Non-Hermitian Morse Case

Let us assume the potential parameters $V_1 = (\alpha + i\beta)^2$ and $V_2 = (2\gamma + 1)(\alpha + i\beta)$ in Eq. (17). Here we choose α and β and $\gamma = -\frac{1}{2} + q(\alpha + i\beta)$. When $a \rightarrow\ i\ a$ in Eq. (17) and choosing V_1 and V_2 as in the previous section, the potential form will be

$$V_M(x) = (\alpha + i\beta)^2(e^{-2iax} - 2qe^{-iax}). \tag{37}$$

The ansatz equation is

$$W_1(x) = \xi\ e^{-ia\ x} + iq\ \delta. \tag{38}$$

As a result the Schrödinger equation can be written by using the Eq. (19) for $\mu^2 = \dfrac{2m(\alpha + i\beta)^2}{a^2\hbar^2}$ and $E = \varepsilon\dfrac{a^2\hbar^2}{2m}$ $(if E < 0)$. Applying the same procedure again, one can get

$$W_{n+1}(x) = \xi\ e^{-ia\ x} + iq\left[i\xi - \frac{a}{q}(n+\frac{1}{2})\right], \tag{39}$$

$$V_{n+1}(x) = \mu^2(e^{-2iax} - 2qe^{-iax}) + 2in\xi\ a\ e^{-iax}, \tag{40}$$

$$E_{n+1}^{(\ell=0)} = -q^2\left[i\xi - \frac{a}{q}(n+\frac{1}{2})\right]^2, \tag{41}$$

with

$$\Psi_0(x) = N\ \exp\ \{-\widetilde{\xi}e^{-iax} + iq\left[i\xi - \frac{a}{q}(n+\frac{1}{2})\right]x\}, \tag{42}$$

where $\widetilde{\mu} = ia\mu$.

4 Conclusions

We have used the PT-symmetric formulation developed recently within non-relativistic quantum mechanics to a more general Morse potential. We have solved the Schrödinger equation in one dimension by applying Hamiltonian hierarchy method within the framework of SUSYQM. We discussed many different complex forms of this potential. Energy eigenvalues and corresponding eigenfunctions are obtained exactly. We also point out that the exact results obtained for the complexified Morse potential may increase the number of interesting applications in the study of different quantum mechanical systems. In the case of $\beta = 0$ in Eq. (41), there is only real spectra, when $\alpha = 0$, otherwise there exists a complex-valued energy spectra. This implies that broken PT-symmetry doesn't occur spontaneously. Moreover, PT-/non-PT-symmetric non-Hermitian solutions have the same spectra. We also note that both real and imaginary part of the energy eigenvalues corresponds to the anharmonic and harmonic oscillator solutions. The $(n+1)-th$ member superpotential, its superpartner and also corresponding ground state eigenfunctions of PT-symmetric non-Hermitian potentials satisfy the condition of PT-invariance though the others are not.

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